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A SEMI ANALITYCAL SOLUTION OF BOUSSINESQ EQUATION USING ASYMPTOTIC METHOD

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Abstact.
This study aims to determine solution of Boussinesq equation which is approximated by using asymptotic expansion method. Nonlinearity of Boussinesq equation causes the solution is not easily determined, so the solution is approached through its linearity. This method is in the form of power series expansion up to third-order, where each term of the series is linear. Furthermore, the finding solution is compared with the solution that was found by Mohyud in previous study. The result of this comparison showed that there were similarities of the two solutions. However, differences occur in phase as modeled in the solution of each method

Keywords: Boussinesq equation, asymptotic expansion method, resonance term

Introduction

The theory of PDEs has long been one of the most important fields in mathematics. It is essentially due to the frequent occurrence equation and the wide range of applications of PDEs in many branches of physics, engineering and other sciences (Myint-u and Debnath, 2007). One of nonlinear PDEs is Boussinesq equation. It is nonlinear PDEs which describes the propagation of long waves in shallow water under gravity propagating in both direction (Wazwaz, 2009).The Boussinesq equation was introduced by Joseph Valentin Boussinesq (1842-1929) in 1872.

$$u_{tt} - u_{xx} - 3(u^2)_{xx} + u_{xxxx} = 0 \qquad (1)$$

The Boussinesq equation used in this study is equation (1). It is called the good Boussinesq equation or well-posed because it only has a single solution (Wazwaz, 2009). Mohyud (2008) in his paper titled "Exp-Fuction Method for Generalized Traveling Solutions of Good Boussinesq Equation" found a solution of this equation by using Exp-function method.

Different from previous studies, in this study, solution of Boussinesq equation will be determined by using the asymptotic expansion method. This method is an expansion in form of the power series against amplitude elevation. Here the asymptotic expansion is limited only to third-order.

Mathematical Model

The 'good' Boussinesq equation which describes the propagation of a long wave in shallow water in both directions has the form

$$\eta_{tt} - \eta_{xx} - 3(\eta^2)_{xx} + \eta_{xxxx} = 0 \qquad (2)$$

with η is elevation of the wave. We approximate the solution of equation (2) by using an expansion up to third order in the power series of the wave elevation following form

$$\eta \approx \varepsilon\eta^{(1)} + \varepsilon^2\eta^{(2)} + \varepsilon^3\eta^{(3)} \qquad (3)$$

with $\eta^{(1)}, \eta^{(2)}$ and $\eta^{(3)}$ describe type of first order, second order and third order while ε is a positive small number representing the order of magnitude of the wave amplitude.

Third Solution of Boussinesq Equation

Asymptotic expansion method is a standard method used in approaching the nonlinear differential equation solution. The solution will be approximated by asymptotic expansion where the amplitude is expanded in the form of power series up to certain order. The higher order is used, then the better the solutions obtained.

By substituting equation (3) to equation (2), it is obtained polynomial in ε form. By collecting all coefficient from the polynomial in each order, the obtained equations for the first order, second order and third order are as follows :

$$o(\varepsilon) \approx \eta_{tt}^{(1)} - \eta_{xx}^{(1)} - \eta_{xxxx}^{(1)} = 0 \tag{4}$$

$$o(\varepsilon^2) \approx \eta_{tt}^{(2)} - \eta_{xx}^{(2)} - \eta_{xxxx}^{(2)} = 3\left(\eta^{(1)}\eta^{(1)}\right)_{xx} \tag{5}$$

$$o(\varepsilon^3) \approx \eta_{tt}^{(3)} - \eta_{xx}^{(3)} - \eta_{xxxx}^{(3)} = 6\left(\eta^{(1)}\eta^{(2)}\right)_{xx} \tag{6}$$

Of three forms of orders above, the first-order is a homogeneous equation, the second and the third-order is a non homogeneous equation. In this paper, we choose $\eta^{(1)}$ as first-order solution in form

$$\eta^{(1)}(x,t) = \tfrac{1}{2}Ae^{(kx-\omega t)} + \tfrac{1}{2}Ae^{-(kx-\omega t)}. \tag{7}$$

By solving the nonhomogeneous terms in the right side of equation (5), it obtain the second-order solution as follows

$$\eta^{(2)} = B_1 e^{2(kx-\omega t)} + B_2 e^{-2(kx-\omega t)} \tag{8}$$

Equation (8) is derivated to variable t and x and subtituted to equation (5), so it obtain the following equation:

$$B_1 = \frac{3k^2A^2}{4\omega^2 - 4k^2 + 16k^4} \text{ and}$$

$$B_2 = \frac{3\,k^2A^2}{4\omega^2 - 4k^2 + 16k^4}.$$

By solving the nonhomogeneous term in the right side of equation (6) it obtain the third-order solution as follows

$$\begin{aligned}\eta^{(3)}(x,t) &= C_1 e^{(kx-\omega t)} + C_2 e^{-(kx-\omega t)} \\ &\quad + C_3 e^{3(kx-\omega t)} + C_4 e^{-3(kx-\omega t)}\end{aligned} \tag{9}$$

Equation (9) is derivated to variable t and x and subtituted to equation (6), so it obtain the following equation

$$C_1 = \frac{[illegible]}{[illegible]}, \qquad C_2 = \frac{[illegible]}{[illegible]}$$

$$C_3 = \frac{[illegible]}{[illegible]}, \qquad C_4 = \frac{[illegible]}{[illegible]}$$

The solution that has been obtained is substituted into equation (3), so that the approximation solution can be written as :

$$\begin{aligned}(x,t) \approx{} & \tfrac{1}{2}\varepsilon Ae^{(kx-\omega t)} + \tfrac{1}{2}\varepsilon Ae^{-(kx-\omega t)} \\ & + \varepsilon^2 B_1 e^{2(kx-\omega t)} + \varepsilon^2 B_2 e^{-2(kx-\omega t)} \\ & + \varepsilon^3 C_1 e^{(kx-\omega t)} + \varepsilon^3 C_2 e^{-(kx-\omega t)} \\ & + \varepsilon^3 C_3 e^{3(kx-\omega t)} + \varepsilon^3 C_4 e^{-3(kx-\omega t)}\end{aligned} \tag{10}$$

Correction of Wave Number

By using asymptotic expansion directly to solve equation (1) it will result a resonance in third-order.

It is characterized by increasing wave amplitude with increasing time as shown in Figure 1. To prevent this resonance, correction is done on the wave number. A technique is executed by doing an expansion not only in the elevation of amplitude but also in the wave number. The expansion of the wave number as the power series in the form of epsilon can be written as follows

$$k \approx k_0 + \varepsilon k_1 + \varepsilon^2 k_2 \tag{11}$$

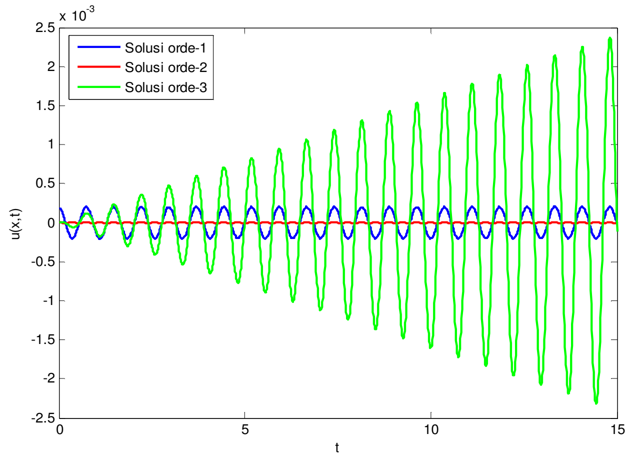


Figure 1. Solutions for each order with $= 0.5$, $x = 3$, $\varepsilon = 0.0001$ and $k = 3$

By Substituting equation (11) to the approximation solution (10) will obtain the following equation

$$\eta(x,t) = \frac{1}{2}\varepsilon A e^{\theta} + \frac{1}{2}\varepsilon A e^{-\theta} + \varepsilon^2 B_1 e^{2\theta} + \varepsilon^2 B_2 e^{-2\theta} + \varepsilon^3 C_1 e^{\theta} + \varepsilon^3 C_2 e^{-\theta} + \varepsilon^3 C_3 e^{3\theta} + \varepsilon^3 C_4 e^{-3\theta} \quad (12)$$

where $\theta = [(k_0 + \varepsilon k_1 + \varepsilon^2 k_2)x - \omega t]$.

Equation (12) is derivated to variable t and x and subtituted to equation (4), equation (5) and equation (6) it obtain the value of $k_1 = 0$ and

$k_2 = \frac{[illegible]}{[illegible]}$

By Substituting value of k_1 and k_2 to the expansion k (11), it obtain following value

$$k \approx k_0 + \varepsilon^2 \left(\frac{6k_0 B}{[illegible]} \right) \quad (13)$$

And then, by eliminating resonant term using expansion k (13), it obtain solution for each order as follows:

$\eta^{(1)} = \frac{1}{2}A e^{[illegible]} + \frac{1}{2}A e^{-[illegible]}$

$\eta^{(2)} = \left([illegible] \right) e^{[illegible]} + \left([illegible] \right) e^{-[illegible]}$

$\eta^{(3)} = \left([illegible] \right) e^{[illegible]} + \left([illegible] \right) e^{-[illegible]}$

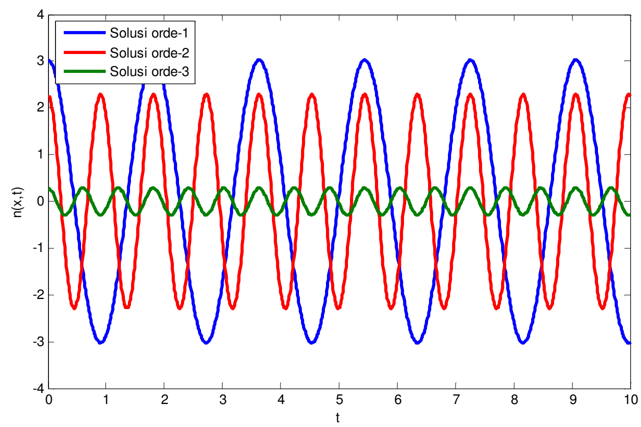


Figure 2. Solution for each order after corrected by k_2 with $A = 1.5$, $x = 3$, $\varepsilon = 0.01$ and $k_0 = 2$

Comparison Between Asymptotic Expansion Solution and Exp-Function Solution

Of all forms of first-order $\eta^{(1)}$, second-order $\eta^{(2)}$, and third-order $\eta^{(3)}$ are substituted to equation (2) so that it generate an approximation solution of Boussinesq Equation that can be written as follows:

$$\begin{aligned}\eta(x,t) = {} & \frac{1}{2}\varepsilon A e^{(kx-\omega t)} + \frac{1}{2}\varepsilon A e^{-(kx-\omega t)} \\ & +\varepsilon^2\left(\frac{[illegible]}{[illegible]}\right)e^{2(kx-\omega t)} \\ & +\varepsilon^2\left(\frac{[illegible]}{[illegible]}\right)e^{-2(kx-\omega t)} \\ & +\varepsilon^3\left(\frac{[illegible]}{[illegible]}\right)e^{3(kx-\omega t)} \\ & +\varepsilon^3\left(\frac{[illegible]}{[illegible]}\right)e^{-3(kx-\omega t)}\end{aligned}$$

with $\omega = \sqrt{k_0^2 - k_0^4}$. Here are the view of the solution for the value of k_0 that satifies $k_0 > 1$ or $0 < k_0 < 1$.

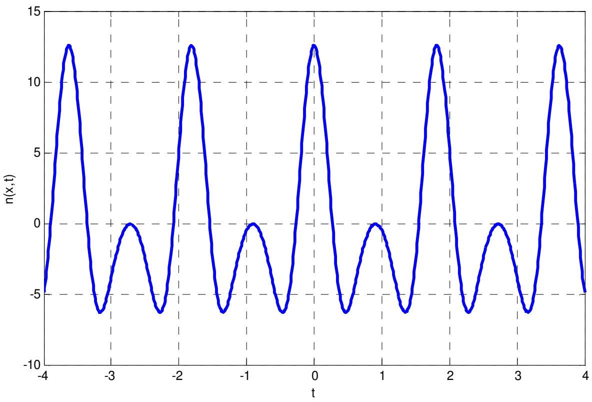


Figure 3. Solution of Boussinesq Equation using asymptotical method, with $A = 2.48$, $k_0 = 2$, $\varepsilon = 0.01$ and $x = 3$

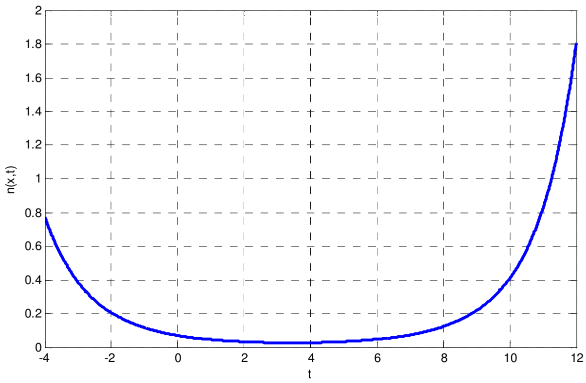


Figure 4. Solution of Boussinesq Equation using asymptotical method, with $A = 2.48$, $k_0 = 0.5$, $\varepsilon = 0.01$ and $x = 3$

Next, the solution that is obtained by asymptotic expansion method is compared with the solution that is obtained by exp-function method. This comparison is executed by plotting both of curve solutions with the parameter values that has been adjusted.

Mohyud (2008) in his study entitled "Exp-Function for Generalized Traveling Solutions of Good Boussinesq Equation" obtained solution of equation (1) as follows

$$\eta(x,t) = \frac{\omega^2 + k^2 - k^4}{2k^2} - \frac{3b_1k^2}{\frac{b_1^2}{b_0}e^{(2kx-2\omega t)} + b_1 e^{(kx-\omega t)} + b_0}$$

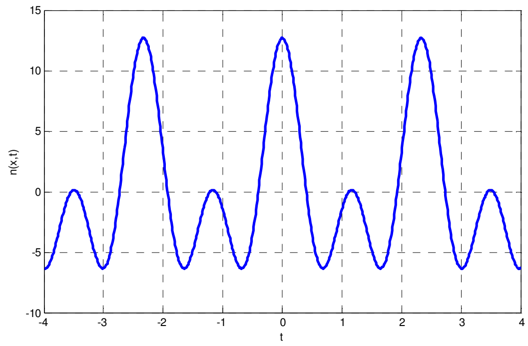


Figure 5. Solution of Boussinesq Equation using Exp-Function method, with $x = 0.9$ and $k = 2$

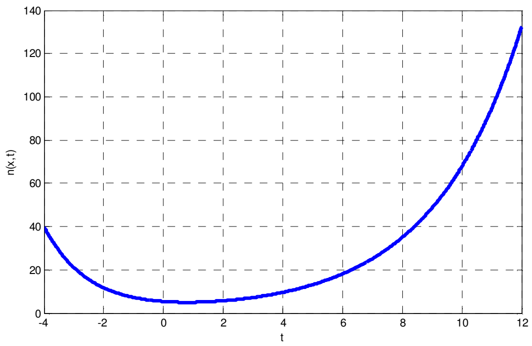


Figure 6. Solution of Boussinesq Equation using Exp-Function method, with $x = 0.9$ and $k = 0.5$

From both Boussinesq solutions with different method above, the relation for the value of k_0 that satifies $k_0 > 1$or $0 < k_0 < 1$ is obtained as follows

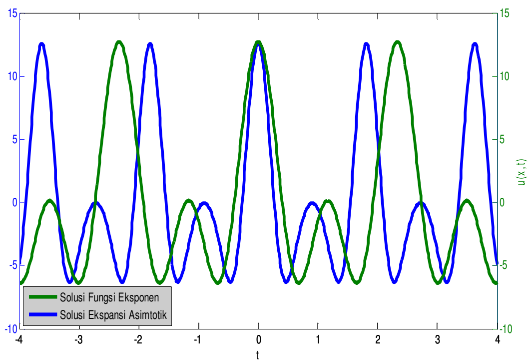


Figure 7. Comparison both of Boussinesq solutions using different method with $\varepsilon = 0.01$, $k_o = 3$, $x = 3$ and $t = [-4:4]$

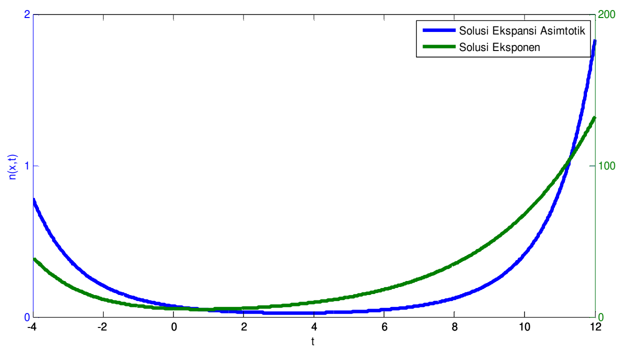
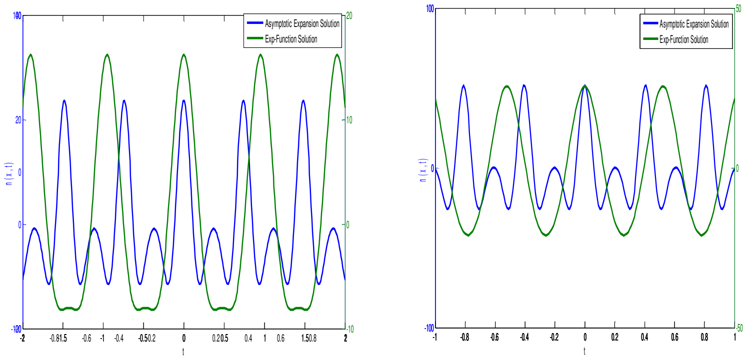


Figure 8. Comparison both of Boussinesq solutions using different method with $\varepsilon = 0.01$, $k_o = 0.5$, $x = 3$ and $t = [-4:4]$

From figure 7 and figure 8, it can be seen that both of Boussinesq solutions with different methods have the same pattern namely wave pattern. For figure 7, when $t = 0$ both peak of solutions coincide at the same point. Both of the solutions look similar but have differences on the phase when it is modeled in each solution.



(a) (b)

Figure 9. Comparison both of Boussinesq solutions using different method with $\varepsilon = 0.001, x = 3$ $t = [-4: 4]$ and (a) $k_0 = 3$ (b) $k_0 = 4$

But for large values of k_0 both of solutions are not similar as shown in figure 9.

Conclusion

Construction solution of Boussinesq Equation is approached by asymptotic expansion method up to third order. In this process there are constraint in the form of resonance. To prevent the resonant term, we expand wave number that corrects it, in this study are third order.

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